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Centre and representations of $\mathcal{U}_q(sl(2|1))$ at roots of unity

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Abstract. Quantum groups at the roots of unity have the property that their centre is enlarged. Polynomial equations relate the standard deformed Casimir operators and the new central elements. These relations are important from a physical point of view, since they correspond to relations between quantum expectation values of observables that have to be satisfied on all physical states. In this paper, we establish these relations in the case of the quantum Lie superalgebra $\mathcal{U}_q(sl(2|1))$. In the course of the argument, we find and use a set of representations such that any relation satisfied on all the representations of the set is true in $\mathcal{U}_q(sl(2|1))$. This set is a subset of the set of all the finite-dimensional irreducible representations of $\mathcal{U}_q(sl(2|1))$, which we classify and describe explicitly.

1. Introduction

Classical and quantum Lie superalgebras and their representations respectively play an important role in the understanding and exploitation of classical and q-deformed supersymmetry in physical systems. A complete classification of the finite-dimensional simple classical Lie superalgebras over $\mathbb{C}$ has been given by Kac [11, 12] and Scheunert [23]. The corresponding irreducible representations fall into two series, called typical and atypical.

Irreducible representations of the quantum analogue of superalgebras are studied intensively when q is not root of unity in [20, 21, 18, 25].

A complete classification of finite-dimensional irreducible representations of unrestricted quantum algebras for q being a root of unity exits only in the particular case of $\mathcal{U}_q(sl(2))$ [22]. Partial classifications exist also in the case of $\mathcal{U}_q(sl(3))$, in [6] for the restricted case and in [2] for periodic representations. Considerable progress towards a complete classification in the general case of $\mathcal{U}_q(\mathcal{G})$ for $\mathcal{G}$ being a simple Lie algebra was made in [4, 5].

The classification of finite-dimensional irreducible representations of $\mathcal{U}_q(osp(1|2))$ for any q parallels the $\mathcal{U}_q(sl(2))$ case [9, 16, 19, 26]. The only other fully understood case is $\mathcal{U}_q(sl(2|1))$ [25, 27].

Our main goal in this paper is the structure of the centre of $\mathcal{U}_q(sl(2|1))$ when q is a root of unity. Complete sets of representations, to be defined below, give a convenient way of

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proving relations in this centre. Their construction involves a detailed knowledge of matrix elements of the finite dimension irreducible representations, whose classification is given below, with emphasis on what is needed for the rest of the paper.

In section 2, we give the definition of the quantum superalgebra $\mathcal{U}_q(sl(2|1))$ and the expression for the central elements. Generalities on the finite-dimensional irreducible representations of $\mathcal{U}_q(sl(2|1))$ are presented in section 3. In section 4, we recall some useful results on $\mathcal{U}_q(gl(2))$ at roots of unity and we give complete sets of irreducible representations for this quantum algebra: expressions in the universal quantum enveloping algebra that vanish on such sets, vanish identically. In section 5, we classify the finite-dimensional irreducible representations of $\mathcal{U}_q(sl(2|1))$. In section 6, we present complete sets of representations corresponding to infinite subsets of the set of continuous parameters. All the representations of these complete sets have the same dimension, unlike the classical case [1]. Finally, in section 7, we prove the relations in the centre using our complete set of irreducible representations.

2. Quantum superalgebra $\mathcal{U}_q(sl(2|1))$ and its centre

The superalgebra $\mathcal{U}_q(sl(2|1))$ is the associative superalgebra over $\mathbb{C}$ with generators $k_1 = q^{h_1}$, $k_1^{-1} = q^{-h_1}$, $k_2 = q^{h_2}$, $k_2^{-1} = q^{-h_2}$, e_1, e_2, f_1, f_2 and relations

$$k_1 k_2 = k_2 k_1 \tag{1}$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j \qquad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j \tag{2}$$

$$e_1 f_1 - f_1 e_1 = \frac{k_1 - k_1^{-1}}{q - q^{-1}} \qquad e_2 f_2 + f_2 e_2 = \frac{k_2 - k_2^{-1}}{q - q^{-1}} \tag{3}$$

$$[e_1, f_2] = 0 \qquad [e_2, f_1] = 0 \tag{4}$$

$$e_2^2 = f_2^2 = 0 \tag{5}$$

$$e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0 \tag{6}$$

$$f_1^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0\,. \tag{7}$$

The last two equations are called the Serre relations. The matrix (a_{ij}) is the distinguished Cartan matrix of $sl(2|1)$, i.e.

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

The $\mathbb{Z}_2$-grading in $\mathcal{U}_q(sl(2|1))$ is uniquely defined by the requirement that the only odd generators are e_2 and f_2, i.e.

$$\begin{aligned} &\deg(k_1) = \deg(k_2) = 0 \\ &\deg(k_1^{-1}) = \deg(k_2^{-1}) = 0 \\ &\deg(e_1) = \deg(f_1) = 0 \\ &\deg(e_2) = \deg(f_2) = 1\,. \end{aligned} \tag{9}$$

We will not use the (standard) co-algebra structure in what follows.

Define

$$e_3 = e_1\, e_2 - q^{-1} e_2\, e_1 \qquad \text{and} \qquad f_3 = f_2\, f_1 - q\, f_1\, f_2\,. \tag{10}$$

The quantum Serre relations become

$$\begin{aligned} e_1\, e_3 &= q\, e_3\, e_1 \\ f_3\, f_1 &= q^{-1} f_1\, f_3\,. \end{aligned} \tag{11}$$

Furthermore

$$\begin{aligned} e_2\, e_3 &= -q\, e_3\, e_2 \\ f_3\, f_2 &= -q^{-1} f_2\, f_3 \end{aligned} \tag{12}$$

and

$$\begin{aligned} e_3\, f_3 + f_3\, e_3 &= \frac{k_1\, k_2 - k_1^{-1}\, k_2^{-1}}{q - q^{-1}} \\ e_3^2 = f_3^2 &= 0\,. \end{aligned} \tag{13}$$

In what follows, we will use the conventional notation

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}\,. \tag{14}$$

When q is not a root of unity, the centre of $\mathcal{U}_q(sl(2|1))$ is generated by the elements $\mathcal{C}_p,\ p \in \mathbb{Z}$, where

$$\begin{aligned} \mathcal{C}_p = k_1^{2p-1} k_2^{4p-2} (q - q^{-1})^2 \Big\{ & [h_1 + h_2 + 1][h_2] - f_1 e_1 \\ & + f_2 e_2 ([h_1 + h_2] q^{1-2p} - [h_1 + h_2 + 1]) \\ & + f_3 e_3 ([h_2 - 2] q^{1-2p} - [h_2 - 1]) \\ & + (q - q^{-1}) q^{-1-p} [p] f_3 e_2 e_1 k_2 + (q - q^{-1}) q^{2-p} f_1 f_2 e_3 k_2^{-1} [p - 1] \\ & + (q - q^{-1})^2 q^{1-2p} [p][p-1] f_2 f_3 e_3 e_2 \Big\}\,. \end{aligned} \tag{15}$$

They satisfy the relations

$$\mathcal{C}_{p_1} \mathcal{C}_{p_2} = \mathcal{C}_{p_3} \mathcal{C}_{p_4} \qquad \text{if} \quad p_1 + p_2 = p_3 + p_4\,. \tag{16}$$

The fact that the centre was not finitely generated in the classical case was discovered in [13, 24]. The explicit expression for a set of generators of the centre, together with the relations, was given in [1] in the classical case and in [3] in the quantum case.

In this paper, we consider the case where q is a root of unity. Let l be the smallest integer such that $q^l = 1$. We define

$$l' = \begin{cases} l & \text{if } l \text{ is odd} \\ l/2 & \text{if } l \text{ is even} \end{cases} \tag{17}$$

the elements $z_i \equiv k_i^l$, $x_1 \equiv e_1^l$ and $y_1 \equiv f_1^l$ also belonging to the centre.

Proposition 1. When l is odd, the central elements $z_1,\ z_2,\ x_1,\ y_1$ and $\mathcal{C}_p,\ p \in \mathbb{Z}$ satisfy the relations

$$\begin{aligned} & \mathcal{C}_{p+l} = z_1^2 z_2^4 \mathcal{C}_p \\ & \mathcal{C}_{p+1}^l = z_1^2 z_2^4 \mathcal{C}_p^l \\ & \mathcal{P}_l(\mathcal{C}_1, \ldots, \mathcal{C}_l) \equiv (\mathcal{C}_1 + 1)^l - 1 + \sum_{\substack{m \geqslant 2 \\ n \geqslant 0 \\ m+n \leqslant l}} \mathcal{C}_m \mathcal{C}_1^n \frac{l}{m-1} \binom{m+n-1}{n+1} \binom{l-m}{n} \\ & \qquad = \left(1 - z_1^2 z_2^2\right)\left(z_2^2 - 1\right) - (q - q^{-1})^{2l} z_1^2 z_2^4 y_1 x_1\,. \end{aligned} \tag{18}$$

The first two relations follow from the expression for $\mathcal{C}_p$ and from (16). The third relation will be proved using complete sets of representations of $\mathcal{U}_q(sl(2|1))$. Furthermore, there is no other independent polynomial relation.

3. Generalities on finite-dimensional irreducible representations

Let us consider a finite-dimensional irreducible left module M over $\mathcal{U}_q(sl(2|1))$.

- The generators k_1 and k_2 are simultaneously diagonalizable on the module M.
- Since $e_2^2 = 0$ and $\dim M < \infty$, there exists a subspace $V \subset M$ annihilated by e_2, i.e.

$$\forall\, v \in V \qquad e_2\, v = 0\,. \tag{19}$$

- Since $e_3^2 = 0$ and $e_2\, e_3 = -q\, e_3\, e_2$, there exists $V_0 \subset V$ annihilated by e_3, i.e.

$$\forall\, v \in V_0 \qquad e_2\, v = e_3\, v = 0\,. \tag{20}$$

- Because of (2) the subspace V_0 is stable by left multiplication by k_1 and k_2.
- Because of (10) and (11) the subspace V_0 is stable by left multiplication by e_1.
- Because of $e_2\, f_1 = f_1\, e_2$ and $e_3\, f_1 - f_1\, e_3 = -e_2\, k_1^{-1}$ the subspace V_0 is stable by left multiplication by f_1.

Let $\mathfrak{g}_{\bar 0} \simeq gl(2)$ be the even subalgebra of $sl(2|1)$. The algebra $\mathcal{U}_q(\mathfrak{g}_{\bar 0})$ is generated by $e_1,\ f_1,\ k_1$ and k_2.

The module V_0 is then a $\mathcal{U}_q(\mathfrak{g}_{\bar 0})$ submodule of M. It is simple (as a $\mathcal{U}_q(\mathfrak{g}_{\bar 0})$ module), since any submodule of V_0 would generate a proper submodule of M by left action of $\mathcal{U}_q(sl(2|1))$. As a consequence of the simplicity of V_0, the element $k_1\, k_2^2$ (the $U(1)$ generator) is represented by a scalar on V_0.

Let $\mathcal{U}_q(\mathfrak{g}_+)$ be the subalgebra of $\mathcal{U}_q(sl(2|1))$ generated by $e_1,\ f_1,\ k_i$ and e_2. The subspace V_0 is also a $\mathcal{U}_q(\mathfrak{g}_+)$-module, annihilated by e_2.

From V_0 considered as an $\mathcal{U}_q(\mathfrak{g}_+)$-module, one can construct an induced $\mathcal{U}_q(sl(2|1))$-module $M' = \mathcal{U}_q(sl(2|1)) \otimes_{\mathcal{U}_q(\mathfrak{g}_+)} V_0$. Then M is equal to M' if M' is simple, or to the quotient of M' by its maximal submodule otherwise.

Since we already know that each finite-dimensional irreducible representation of $\mathcal{U}_q(sl(2|1))$ is associated with one finite-dimensional irreducible representation V_0 of $\mathcal{U}_q(\mathfrak{g}_{\bar 0})$, we will construct the classification of the former in terms of the latter. As we will see, the correspondence is one-to-one. We now need some results on $\mathcal{U}_q(gl(2))$ at roots of unity.

4. $\mathcal{U}_q(gl(2))$ at roots of unity

4.1. The centre of $\mathcal{U}_q(gl(2))$

The elements $k_1k_2^2,\ k_1^{l'},\ k_2^{l},\ e_1^{l},\ f_1^{l}\ e_1^{l'} f_1^{l'},\ f_1^{l'} e_1^{l'}$ are central in $\mathcal{U}_q(gl(2))$. The q-deformed quadratic Casimir operator is

$$\mathcal{C}_{\mathcal{U}_q(gl(2))} = qk_1 + q^{-1}k_1^{-1} + (q - q^{-1})^2 f_1 e_1\,. \tag{21}$$

When l is odd, the centre of $\mathcal{U}_q(gl(2))$ is actually the algebra defined by the generators $k_1k_2^2,\ k_1^l,\ e_1^l,\ f_1^l,\ \mathcal{C}_{\mathcal{U}_q(gl(2))}$ and the relation [15]

$$2P_l\left(\mathcal{C}_{\mathcal{U}_q(gl(2))}/2\right) = k_1^l + k_1^{-l} + (q - q^{-1})^{2l} f_1^l e_1^l\,. \tag{22}$$

The polynomial P_l is the first kind Chebyshev polynomial of degree l defined by

$$P_l(\cos x) = \cos(lx)\,. \tag{23}$$

4.2. Finite-dimensional irreducible representations of $\mathcal{U}_q(gl(2))$

All the finite-dimensional simple modules over $\mathcal{U}_q(gl(2))$ are of course cyclic. We denote those representations that are deformations of the classical representations *type* $\mathcal{A}$, and the others *type* $\mathcal{B}$. Knowing the $\mathcal{U}_q(sl(2))$ case, we only need to add a parameter related to the value of the $U(1)$ generator $k_1k_2^2$. This parameter may be provided as $\lambda_1\lambda_2^2$ (value of $k_1k_2^2$) and a sign, or simply by the value λ_2 of k_2 on a given vector. The finite-dimensional irreducible representations of $\mathcal{U}_q(sl(2))$ are [22]:

- *type* $\mathcal{A}$: usual (nilpotent) representations, where $k_1^{2l} = 1$, $e_1^{l'} = 0$, $f_1^{l'} = 0$, characterized by their dimension $N = 1, \ldots, l'$ and a sign ω. The highest weight λ_1 is $\lambda_1 = q^{N-1}$. These representations are given explicitly in equation (A1). The representation of dimension l' plays a special role. It is in fact in the intersection of this case and the following.
- *type* $\mathcal{B}$: coloured (nilpotent) representations, with still $e_1^{l'} = 0$, $f_1^{l'} = 0$, characterized by their highest weight λ_1, a continuous parameter. Their dimension is l'. They are also described by (A1).
- *type* $\mathcal{B}$: periodic and semi-periodic representations, explicitly given in (A2). These representations have dimension l. They depend on four complex parameters corresponding to the values of the three central elements $k_1^{l'}$, e_1^l, f_1^l and one discrete parameter corresponding to the value of the quadratic Casimir $\mathcal{C}_{\mathcal{U}_q(gl(2))}$ of $\mathcal{U}_q(gl(2))$ and related to the former through the relation (22). The $\mathcal{U}_q(gl(2))$-representation is also completely characterized by the parameters $y = \varphi^l$, β, λ_1 and λ_2 appearing in

$$\begin{aligned} &f_1^l v_0 = \varphi^l v_0 && f_1 e_1 v_0 = \beta v_0 \\ &k_1 v_0 = \lambda_1 v_0 = q^{\mu_1} v_0 && k_2 v_0 = \lambda_2 v_0 = q^{\mu_2} v_0. \end{aligned} \tag{24}$$

The existence of periodic irreducible representations has the following consequence: the primitive ideals defined as the kernels of these representations are not the annihilator of the irreducible quotient of some Verma module, unlike the case of classical (super)algebras [8, 17].

4.3. Complete sets of representations of $\mathcal{U}_q(sl(2))$

We prove that a set of generic (periodic) representations corresponding to an open subset of the set of parameters builds a complete set, in the following sense: if an element of $\mathcal{U}_q(sl(2))$ acts as 0 on all the representations of this set, then it is the 0 element of $\mathcal{U}_q(sl(2))$. This terminology was used in [1], where the authors found complete sets of finite-dimensional irreducible representations of the classical $sl(2)$ and $sl(2|1)$. For quantum groups at roots of unity, we shall obtain rather different results.

Let $\mathcal{R} \in \mathcal{U}_q(sl(2))$ be such that it vanishes on a set Ω of representations. Let $q^{2(t-r)}$ be the q-grading of an element $f_1^r k_1^s e_1^t$. We have $k_1\left(f_1^r k_1^s e_1^t\right) = q^{2(t-r)}\left(f_1^r k_1^s e_1^t\right)k_1$. Any element of $\mathcal{U}_q(sl(2))$ is a sum of terms of given grading since the $f_1^r k_1^s e_1^t$ form a basis of $\mathcal{U}_q(sl(2))$. We write $\mathcal{R} = \sum_{d=0}^{l'-1} \mathcal{R}_d$, where the grading of $\mathcal{R}_d$ is q^{2d}. Commuting $\mathcal{R}$ with k_1 shows that all the $\mathcal{R}_d$ vanish separately on the representations in Ω. The same is true for each $f_1^d \mathcal{R}_d$. Since $\mathcal{U}_q(sl(2))$ contains no zero divisor [14], the vanishing of $f_1^d \mathcal{R}_d$ in $\mathcal{U}_q(sl(2))$ is equivalent to that of $\mathcal{R}_d$. Hence, to prove that Ω is a complete set of representations, we only have to show that the only element of $\mathcal{U}_q(sl(2))$ commuting with k_1 and acting as 0 on all representations of Ω is 0.

Let $\mathcal{R}$ be an element of $\mathcal{U}_q(sl(2))$ with grading 1, and $n, n' \in \mathbb{N}$ such that $f_1^n k_1^{n'} \mathcal{R} = \sum_i a_i f_1^{r_i} k_1^{s_i} e_1^{t_i}$ has only terms with $r_i - t_i \in l'\mathbb{N}$ and $s_i \in \mathbb{N}$. Then $f_1^n k_1^{n'} \mathcal{R}$ can be written as a polynomial in $f_1^{l'}$, k_1, and $f_1 e_1$, which commute with each other. The value of this polynomial on the vector v_0 of the representation (A2) is the same polynomial evaluated on the scalars $\varphi^{l'}$, λ_1 and β. If Ω is a set of representations corresponding to an open subset of $\mathbb{C}^3$ for the values of $\varphi^{l'}$, λ_1 and β, and if $\mathcal{R}$ vanishes on all the representations of Ω, then the polynomial vanishes identically in $\mathcal{U}_q(sl(2))$, and hence $\mathcal{R} = 0$ as an element of $\mathcal{U}_q(sl(2))$. We then have the following proposition.

Proposition 2. A set of generic (periodic) representations corresponding to an open subset of the set of values for the parameters is a complete set of representations.

Remark 1. An element of $\mathcal{U}_q(sl(2))$ that vanishes on all *type* $\mathcal{A}$ modules, or even on all nilpotent or semiperiodic modules, is not necessarily 0 in $\mathcal{U}_q(sl(2))$ (take simply $\mathcal{R} = e_1^l f_1^l$). So a complete set of representations should include periodic ones.

Remark 2. Suitably choosen infinite sets of periodic representations (not necessarily corresponding to an open set of values of the parameters) can also be complete.

5. Classification of finite-dimensional irreducible representations of $\mathcal{U}(sl(2|1))$

Let V_0 an N-dimensional irreducible $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-module, that we extend to a $\mathcal{U}_q(\mathfrak{g}_+)$-module by the requirement that $e_2 V_0 = 0$.

Let M' be the induced module $\mathcal{U}_q(sl(2|1)) \otimes_{\mathcal{U}_q(\mathfrak{g}_+)} V_0$. Then

$$M' = V_0 \oplus f_2 V_0 \oplus f_3 V_0 \oplus f_2 f_3 V_0 \,. \tag{25}$$

The subspaces $(f_2 V_0 \oplus f_3 V_0)$ and $f_2 f_3 V_0$ are representations of $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$ with the same value for central elements $k_1^{l'}$, k_2^l, e_1^l, f_1^l as for V_0. If we write the value of quadratic Casimir $\mathcal{C}_{\mathcal{U}_q(gl(2))}$ of $\mathcal{U}_q(gl(2))$ as $\xi + \xi^{-1}$, then its eigenvalues on the different subspaces are

$$\begin{array}{lll} \text{Subspace} & \mathcal{C}_{\mathcal{U}_q(gl(2))} & \\ & & \\ V_0 : & \xi + \xi^{-1} & \\ (f_2 V_0 \oplus f_3 V_0) : & q\xi + q^{-1}\xi^{-1}, & q^{-1}\xi + q\xi^{-1} \\ f_2 f_3 V_0 : & \xi + \xi^{-1} \,. & \end{array} \tag{26}$$

The elements $f_2^\rho f_3^\sigma f_1^p$, for $p \in \mathbb{N}$, $\rho \in \{0, 1\}$ and $\sigma \in \{0, 1\}$ build a Poincaré–Birkhoff–Witt basis of the subalgebra $\mathcal{U}^-$ generated by f_1 and f_2. The elements $e_1^{p'} e_3^{\sigma'} e_2^{\rho'}$, for $p' \in \mathbb{N}$, $\rho' \in \{0, 1\}$ and $\sigma' \in \{0, 1\}$ build a Poincaré–Birkhoff–Witt basis of the subalgebra $\mathcal{U}^+$ generated by e_1 and e_2. Together with the basis $k_1^{s_1} k_2^{s_2}$ (with $s_i \in \mathbb{Z}$) for the Cartan subalgebra, this provide a basis for $\mathcal{U}_q(sl(2|1))$.

Let $w_{0,0,0}$, $w_{0,0,1}, \ldots, w_{0,0,N-1}$, be a basis of V_0. Then it follows from the definition of V_0 and of the Poincaré–Birkhoff–Witt basis of $\mathcal{U}_q(sl(2|1)$ given above that the vectors $f_2^\rho f_3^\sigma w_{0,0,p}$ $\rho, \sigma \in \{0, 1\}$, $p \in \{0, N-1\}$ build a basis of M'. In particular

$$\dim M' = 4N \tag{27}$$

i.e. four times the dimension of V_0.

Since the dimension N of V_0 is bounded by l, we already know that the dimension of a simple $\mathcal{U}_q(sl(2|1))$-module is bounded by $4l$. Since nilpotent representations of $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$ have dimension less or equal to l', the dimension of nilpotent representations of $\mathcal{U}_q(sl(2|1))$ is bounded by $4l'$.

5.1. Usual (type $\mathcal{A}$) representations

We now start from a $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-module V_0 which is the q-deformation of a classical module. Let N be its dimension ($1 \leqslant N \leqslant l'$).

The module M' is then a highest weight module with highest weight vector $w_{0,0,0}$ on which

$$\begin{array}{ll} e_1\, w_{0,0,0} = 0 & e_2\, w_{0,0,0} = 0 \\ k_1\, w_{0,0,0} = \lambda_1 w_{0,0,0} & k_2\, w_{0,0,0} = \lambda_2 w_{0,0,0} \end{array} \tag{28}$$

with $\lambda_1 = \omega q^{N-1}$, $\omega = \pm 1$.

The Casimir operators $\mathcal{C}_p$ have the following scalar value on M':

$$\mathcal{C}_p = (q - q^{-1})^2 \lambda_1^{2p-1} \lambda_2^{4p-2} [\mu_2][\mu_1 + \mu_2 + 1] \tag{29}$$

where, again, $q^{\mu_i} \equiv \lambda_i$.

A basis of M' is given by

$$w_{\rho,\sigma,p} = f_2^\rho f_3^\sigma f_1^p w_{0,0,0} \qquad \text{with} \quad \begin{cases} \rho, \sigma \in \{0, 1\} \\ p \in \{0, \ldots, N-1\}\,. \end{cases} \tag{30}$$

By convention, we set

$$w_{\rho,\sigma,N} \equiv 0\,. \tag{31}$$

A non-zero vector in a representation is called singular if it is annihilated by e_1 and e_2 and is contained in a proper subrepresentation. Any submodule of M' contains a singular vector for M'. Indeed, any submodule of M' has its own $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-submodule annihilated by e_2. This last module is also of *type* $\mathcal{A}$ because this property is determined by the scalar value of the central elements, which are determined by V_0. The module M' is simple if, and only if, it contains no singular vector $v_s \neq 0$.

Lemma 1. The non-vanishing of the Casimir operators $\mathcal{C}_p$ is a sufficient condition for M' to be simple.

The comparison of the values of the Casimir operators on the highest weight vector and on the singular vector indeed shows that

$$[\mu_2][N + \mu_2] = 0 \tag{32}$$

is a necessary condition for the existence of a singular vector (which cannot be in V_0 since V_0 is a simple $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-module). This condition amounts to the vanishing of all the Casimir operators $\mathcal{C}_p$. We shall see that this is actually a necessary and sufficient condition for the simplicity of M'.

5.1.1. Typical type $\mathcal{A}$ representations.

Proposition 3. If equation (32) is not satisfied, the module M' is simple. It has dimension $4N$. Its explicit expression is given in (B2). It is called typical.

Proof. If equation (32) is not satisfied, M' contains no singular vector. For $N = 1, \ldots, l' - 1$, the subspace $f_2 V_0 \oplus f_3 V_0$ is the direct sum of $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-modules characterized by the dimensions $N \pm 1$ and sign ω. For $N = l'$, $f_2 V_0 \oplus f_3 V_0$ is an indecomposable $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-module which is isomorphic to the tensor product of V_0 with the spin-$\frac{1}{2}$ representation, and which contains the dim $= l' - 1$ (sign $= \omega$) simple sub-$\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-module. □

5.1.2. Atypical type $\mathcal{A}$ representations. We now consider the case $[\mu_2][N+\mu_2]=0$ (i.e. $(\lambda_2^2-1)(\lambda_2^2-q^{-2N})=0$). We will prove the following proposition.

Proposition 4. If the Casimir operators $\mathcal{C}_p$ vanish on M', there exists a maximal submodule M'' of M'. The quotient space $M=M'/M''$ is a simple module, called atypical. We can consider three cases:

- If $[\mu_2]=0$ and $[N+\mu_2]\neq 0$, then $\dim M=2N-1$.
- If $[\mu_2]\neq 0$ and $[N+\mu_2]=0$, then $\dim M=2N+1$.
- If $[\mu_2]=0$ and $[N+\mu_2]=0$ (and hence $N=l'$), then $\dim M=2l'-1$.

Proof. Atypical type $\mathcal{A}$ representations with $[\mu_2]=0$ and $[N+\mu_2]\neq 0$. In this case, the vector $f_2w_{0,0,0}=w_{1,0,0}$ is a singular vector. The action of $\mathcal{U}_q(sl(2|1))$ on it generates a $(2N+1)$-dimensional submodule M'' spanned by

$$\begin{aligned} &f_1^p w_{1,0,0}=q^{-p}w_{1,0,p}-q^{-1}[p]w_{0,1,p-1} && p=0,\ldots,N\\ &f_1^p f_3 w_{1,0,0}=-q^{-1}w_{1,1,p} && p=0,\ldots,N-1\,. \end{aligned} \tag{33}$$

This submodule is maximal. Quotienting M' by M'' provides a $(2N-1)$-dimensional simple module M, the expression of which is given in (B3).

Atypical type $\mathcal{A}$ representations with $[\mu_2]\neq 0$ and $[N+\mu_2]=0$. Looking by direct computation for a singular vector, we see that $N=1$, $[1+\mu_2]=0$ is a particular case: it is the only case of existence of a singular vector in $f_2f_3V_0$ (one-dimensional in this case). The singular vector is $w_{1,1,0}=f_2f_3w_{0,0,0}$. It generates only $f_2f_3V_0$ as $\mathcal{U}_q(sl(2|1))$-submodule. The quotient $M'/f_2f_3V_0$ is three-dimensional. It is actually the q-deformed three-dimensional atypical fundamental representation.

If $N\in\{2,\ldots,l'-1\}$, there is a singular vector given by

$$v_s=\lambda_1 q w_{1,0,1}+[\mu_1]w_{0,1,0}\,. \tag{34}$$

It generates the $(2N-1)$-dimensional maximal submodule M'' spanned by

$$\begin{aligned} &f_1^p v_s=\lambda_1 q^{1-p}w_{1,0,p+1}+[\mu_1-p]w_{0,1,p} && p=0,\ldots,N-2\\ &f_1^p f_2 v_s=[\mu_1]w_{1,1,p} && p=0,\ldots,N-1\,. \end{aligned} \tag{35}$$

The quotient $M=M'/M''$ is a $(2N+1)$-dimensional simple module given explicitly in (B4).

Atypical type $\mathcal{A}$ representations with $N=l'$. If $[\mu_2]=0$ and $N=l'$, the vector $w_{1,0,0}$ is singular. The submodule it generates is similar to (33), except that now $f_1^{l'}w_{1,0,0}=0$. However, the vector $w_{0,1,l'-1}$ is subsingular, i.e. its image by e_1 and e_2 is contained is the submodule generated by $w_{1,0,0}$. It belongs to the maximal submodule M'' of M'. Note that $f_1w_{1,0,0}\in M''$ is also singular. The submodule M'' has dimension $2l'+1$ and $M=M'/M''$ has dimension $2l'-1$. It is also described by (B3). □

5.2. Nilpotent type $\mathcal{B}$ representations

We now consider the case where V_0 is a *type* $\mathcal{B}$ nilpotent $\mathcal{U}_q(\mathfrak{g}_{\bar 0})$-module, of dimension $N=l'$, with two parameters λ_1 and λ_2. We assume $[\mu_1+1]\neq 0$ since this case was treated as *type* $\mathcal{A}$. As in the *type* $\mathcal{A}$ case we consider the induced module M', on which (28) applies. A basis for M' is also given by (30) with $N=l'$. We also have

Proposition 5. Nilpotent *type* $\mathcal{B}$ representations fall into two classes

- If $[\mu_2][\mu_1+\mu_2+1] \neq 0$, i.e. $\mathcal{C}_p \neq 0$, then M' is simple. Its dimension is $4l'$ and the parameters are λ_1 and λ_2. Its explicit expression is given in (B2) (typical case).
- If $[\mu_2][\mu_1+\mu_2+1] = 0$, i.e. $\mathcal{C}_p = 0$, then M' has a maximal submodule M'' of dimension $2l'$. Then $M = M'/M''$ has dimension $2l'$ (atypical case).

Proof. As in the *type* $\mathcal{A}$ case, there is no singular vector if the $\mathcal{C}_p$ do not vanish. Now suppose that $[\mu_2][\mu_1+\mu_2+1] = 0$ We can separate this case into two subcases, according to which term of the product vanishes (both terms cannot vanish simultaneously, since $[\mu_1 - p + 1] \neq 0$ for any integer p in *type* $\mathcal{B}$ $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-modules).

- If $[\mu_2] = 0$, the vector $w_{1,0,0}$ is singular. It generates the submodule M'' given as in (33) with $N = l'$, except that now $f_1^{l'} w_{1,0,0} = 0$. Then $\dim M'' = 2l'$. The quotient module hence has dimension $2l'$. It is described by (B3).
- If $[\mu_1+\mu_2+1] = 0$, then there is a singular vector given by (34). It generates the submodule M'' given as in (35) with $N = l'$, except that now $f_1^{l'-1} v_s \neq 0$ also belongs to M'', so that $\dim M'' = 2l'$. Again, $\dim M = 2l'$ and M is described by (B4). □

5.3. Periodic and semi-periodic type $\mathcal{B}$ representations

Let us now consider the case when V_0 is a periodic or semi-periodic $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-module, i.e. with non-vanishing (scalar) value of the central element f_1^l

$$f_1^l = \varphi^l \,\mathrm{id}\,. \tag{36}$$

In $\mathcal{U}_q(sl(2|1))$, f_1^l is also central, so equation (36) also holds in M'.

The value of the central element e_1^l will be a free parameter (possibly zero for semi-periodic representations). One would get the representations with a vanishing value for f_1^l and a non-vanishing value for e_1^l, using the automorphism of $\mathcal{U}_q(sl(2|1))$ given by

$$\begin{aligned} \psi(e_i) &= f_i \qquad & \psi(f_i) &= e_i \\ \psi(k_1) &= k_1^{-1} \qquad & \psi(k_2) &= -k_2^{-1}\,. \end{aligned} \tag{37}$$

The module M' is actually characterized by the following actions on a vector $w_{0,0,0}$ of V_0:

$$\begin{aligned} f_1^l w_{0,0,0} &= \varphi^l w_{0,0,0} \qquad & f_1 e_1 w_{0,0,0} &= \beta w_{0,0,0} \\ k_1 w_{0,0,0} &= \lambda_1 w_{0,0,0} = q^{\mu_1} w_{0,0,0} \qquad & k_2 w_{0,0,0} &= \lambda_2 w_{0,0,0} = q^{\mu_2} w_{0,0,0}\,. \end{aligned} \tag{38}$$

Those values determine the values of e_1^l (using equation (22)) and of $\mathcal{C}_p$:

$$\mathcal{C}_p = (q-q^{-1})^2 \lambda_1^{2p-1} \lambda_2^{4p-2} \left([\mu_2][\mu_1+\mu_2+1] - \beta\right)\,. \tag{39}$$

A basis of M' is given by

$$w_{\rho,\sigma,p} \equiv \varphi^{-\sigma-p} f_2^\rho f_3^\sigma f_1^p w_{0,0,p} \qquad \text{with} \quad \begin{cases} \rho,\sigma \in \{0,1\} \\ p \in \{0,\dots,l-1\}\,. \end{cases} \tag{40}$$

Proposition 6. For periodic and semi-periodic representations, the following alternative holds:

- *(i).* If $[\mu_2][\mu_1+\mu_2+1] - \beta \neq 0$, the module M' is irreducible and its dimension is equal to $4l$. It is described explicitly in equation (B5).
- *(ii).* If $[\mu_2][\mu_1+\mu_2+1] - \beta = 0$, the module M' is not simple. It has a submodule M'' of dimension $2l$ and the factor space M'/M'' is an irreducible module of dimension $2l$, explicitly given by equation (B6).

The cases (ii) corresponds to atypical periodic representations and $[\mu_2][\mu_1+\mu_2+1]=\beta$ is the condition for the vanishing of the Casimir operators $\mathcal{C}_p$ on M'.

Proof. By direct computation, we check that $[\mu_2][\mu_1+\mu_2+1]-\beta=0$ is the necessary and sufficient condition for the existence of a vector (not belonging to V_0), annihilated by both e_2 and e_3. This vector then belongs to $f_2V_0\oplus f_3V_0$ and it generates a $2l$-dimensional subspace spanned by the vectors $w_{1,1,p}$ and $[\mu_2+p+1]w_{0,1,p}-\lambda_2^{-1}q^{-p}w_{1,0,p+1}$ for $p\in\{0,\ldots,l-1\}$. The quotient of M' by this submodule is simple.

6. Complete sets of representations of $\mathcal{U}_q(sl(2|1))$

Proposition 7. A set of typical periodic representations corresponding to an open subset of the set of values of the parameters is a complete set of representations.

Proof. Let Ω be a set of representations, and $\mathcal{R}\in\mathcal{U}_q(sl(2|1))$ such that $\mathcal{R}$ vanishes on all the representations of Ω. As for $\mathcal{U}_q(sl(2))$, we can restrict ourselves to the case where $k_i\mathcal{R}k_i^{-1}=q^{d_i}\mathcal{R}$ for given gradings d_i $(i=1,2)$.

We have in fact to consider five cases, according to the possible gradings with respect to $k_1k_2^2$. All the possible values for d_1+2d_2 are actually $-2,-1,0,1,2$ (this is due to the fact that the squares of fermionic generators vanish, and it can also be read from the Poincaré–Birkhoff–Witt basis).

$$\begin{aligned}
&d_1+2d_2=-2 &&\mathcal{R}^{(-2)}=\mathcal{R}_1e_3e_2\\
&d_1+2d_2=-1 &&\mathcal{R}^{(-1)}=\mathcal{R}_2e_2+\mathcal{R}_3e_3+\mathcal{R}_4f_2e_3e_2+\mathcal{R}_5f_3e_3e_2\\
&d_1+2d_2=0 &&\mathcal{R}^{(0)}=\mathcal{R}_6+\mathcal{R}_7f_2e_2+\mathcal{R}_8f_3e_2+\mathcal{R}_9f_2e_3+\mathcal{R}_{10}f_3e_3+\mathcal{R}_{11}f_2f_3e_3e_2\\
&d_1+2d_2=1 &&\mathcal{R}^{(1)}=\mathcal{R}_{12}f_2+\mathcal{R}_{13}f_3+\mathcal{R}_{14}f_2f_3e_2+\mathcal{R}_{15}f_2f_3e_3\\
&d_1+2d_2=2 &&\mathcal{R}^{(2)}=\mathcal{R}_{16}f_2f_3
\end{aligned}\tag{41}$$

where the $\mathcal{R}_i$ are elements of $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$. We have to prove that all of them vanish. Since Ω is a set of representations corresponding to an open subset of the set of values of the parameters, the representations of $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$ given by the corresponding V_0 is a complete set. If we identify V_0 and $f_2f_3V_0$ (as $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-modules), we see that the vanishing of $\mathcal{R}_1$ and $\mathcal{R}_{16}$ results from this. Let us now consider $\mathcal{R}^{(0)}$, the cases of $\mathcal{R}^{(-1)}$ and $\mathcal{R}^{(1)}$ being simpler. Since $\mathcal{R}^{(0)}e_3e_2=\mathcal{R}_6e_3e_2$ act as zero on all the representations of Ω, then $\mathcal{R}_6=0$. Now, $\mathcal{R}^{(0)}e_2=(\mathcal{R}_9f_2+\mathcal{R}_{10}f_3)e_3e_2$. This operator sends $f_2f_3V_0$ into $f_2V_0\oplus f_3V_0$ and is supposed to act as zero. Looking at the explicit action of this operator on the vector $v_{1,1,p}$ and using the fact that $f_2V_0\oplus f_3V_0$ is generically a direct sum of two inequivalent $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$-modules, we learn that $\mathcal{R}_9=\mathcal{R}_{10}=0$. Multiplying $\mathcal{R}^{(0)}$ on the right by e_3, we then prove in a similar way that $\mathcal{R}_7=\mathcal{R}_8=0$. Finally, the proof that $\mathcal{R}_{11}=0$ mimics the proof of proposition 2. □

7. Proof of the relation in the centre

We now use a complete set of representation to prove the relation

$$\begin{aligned}
\mathcal{P}_l(\mathcal{C}_1,\ldots,\mathcal{C}_l)&\equiv(\mathcal{C}_1+1)^l-1+\sum_{\substack{m\geqslant 2\\ n\geqslant 0\\ m+n\leqslant l}}\mathcal{C}_m\mathcal{C}_1^n\frac{l}{m-1}\binom{m+n-1}{n+1}\binom{l-m}{n}\\
&=\left(1-z_1^2z_2^2\right)\left(z_2^2-1\right)-(q-q^{-1})^{2l}z_1^2z_2^4y_1x_1\,.
\end{aligned}\tag{42}$$

On a typical *type* $\mathcal{B}$ periodic representation characterized by the parameters λ_1, λ_2, φ^l and β, the value of $\mathcal{C}_p$ is

$$\begin{aligned}\mathcal{C}_p = \left(\lambda_1\lambda_2^2\right)^{2p-2}\mathcal{C}_1 \\ &= \lambda_1^{2p-1}\lambda_2^{4p-2}\left((q\lambda_1\lambda_2 - q^{-1}\lambda_1^{-1}\lambda_2^{-1})(\lambda_2 - \lambda_2^{-1}) - (q-q^{-1})^2\beta\right) \\ &= \lambda_1^{2p-1}\lambda_2^{4p-2}\left(q\lambda_1\lambda_2^2 + q^{-1}\lambda_1^{-1}\lambda_2^{-2} - (\xi+\xi^{-1})\right) \\ &= \lambda_1^{2p-1}\lambda_2^{4p-2}\left(q^{1/2}\lambda_1^{1/2}\lambda_2\xi^{1/2} - q^{-1/2}\lambda_1^{-1/2}\lambda_2^{-1}\xi^{-1/2}\right) \\ &\quad\times\left(q^{1/2}\lambda_1^{1/2}\lambda_2\xi^{-1/2} - q^{-1/2}\lambda_1^{-1/2}\lambda_2^{-1}\xi^{1/2}\right) \qquad (43)\end{aligned}$$

where $(q-q^{-1})^{-2}(\xi+\xi^{-1}) \equiv (q-q^{-1})^{-2}(q\lambda_1 + q^{-1}\lambda_1^{-1}) + \beta$ is the value of the $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$ quadratic Casimir operator on the subspace V_0.

The polynomial $\mathcal{P}_l$ in (18) is such that, if we set

$$\begin{aligned}&\mathcal{C}_1 = \lambda_1\lambda_2^2(x_1 - x_1^{-1})(x_2 - x_2^{-1}) \\ &\frac{x_2}{x_1} = \lambda_1\lambda_2^2\end{aligned} \qquad (44)$$

then

$$\mathcal{P}(\mathcal{C}_1,\dots,\mathcal{C}_l) = \lambda_1^l\lambda_2^{2l}\left(x_1^l - x_1^{-l}\right)\left(x_2^l - x_2^{-l}\right) \qquad (45)$$

so that

$$\mathcal{P}(\mathcal{C}_1,\dots,\mathcal{C}_l) = \lambda_1^l\lambda_2^{2l}\left(\lambda_1^l\lambda_2^{2l} + \lambda_1^{-l}\lambda_2^{-2l} - (\xi^l + \xi^{-l})\right). \qquad (46)$$

Using the polynomial relation (22) in $\mathcal{U}_q(\mathfrak{g}_{\bar{0}})$, we identify $(\xi^l + \xi^{-l})$ with the value of $(q-q^{-1})^{2l}f_1^le_1^l + \left(k^l + k^{-l}\right)$ and we get the evaluation of the right-hand side of (18) on the representation. Since this is true for any typical periodic representations, and since the set of those representations is complete, the relation is true in the enveloping algebra.

The existence of any other independent polynomial relation in the centre would imply more relations between the parameters of the periodic representations, so we also conclude that there is no other independent relation.

Appendix A. Finite-dimensional irreducible representations of $\mathcal{U}_q(gl(2))$

Nilpotent modules of $\mathcal{U}_q(gl(2))$

$$\begin{aligned}&k_1v_p = \lambda_1q^{-2p}v_p && \text{for } p\in\{0,\dots,N-1\} \\ &f_1v_p = v_{p+1} && \text{for } p\in\{0,\dots,N-2\} \text{ and } f_1v_{N-1}=0 \\ &e_1v_p = [p][\mu_1 - p + 1]v_{p-1} && q^{\mu_1}\equiv\lambda_1 \\ &k_2v_p = \lambda_2q^pv_p && \text{for } p\in\{0,\dots,N-1\}.\end{aligned} \qquad (A1)$$

The dimension N is the smallest non-negative integer satisfying $[N][\mu_1 - N + 1] = 0$. For usual *type* $\mathcal{A}$ representations, $N \in \{1,\dots,l'\}$ and the highest weight is related to N by $\lambda_1 = \omega q^{N-1}$, with $\omega = \pm 1$.

For nilpotent *type* $\mathcal{B}$ representations $N = l'$ and λ_1 is a free parameter.

If $N = l'$ and $\lambda_1 = \pm q^{-1}$, the representation is still the q-deformation of a classical one, but it has q-dimension $[N] = 0$. This case plays a special role.

Periodic and semi-periodic modules of $\mathcal{U}_q(gl(2))$

$$\begin{aligned} k_1 v_p &= \lambda_1 q^{-2p} v_p \\ f_1 v_p &= \varphi v_{p+1} \\ e_1 v_p &= \varphi^{-1}([p][\mu_1 - p + 1] + \beta) v_{p-1} \\ k_2 v_p &= \lambda_2 q^p v_p \end{aligned} \tag{A2}$$

with $p \in \{0, \ldots, l-1\}$, and $q^{\mu_i} \equiv \lambda_i$, without defining μ_i itself. These representations have no classical analogue (*type* $\mathcal{B}$).

Appendix B. Finite-dimensional irreducible representations of $\mathcal{U}_q(sl(2|1))$

The following relations are used to determine the action of the generators on the representations:

$$\begin{aligned} &f_1\, f_2^\rho f_3^\sigma f_1^p = q^{\sigma-\rho} f_2^\rho f_3^\sigma f_1^{p+1} - \rho(1-\sigma) q^{-\rho} f_2^{\rho-1} f_3^{\sigma+1} f_1^p \\ &f_2\, f_2^\rho f_3^\sigma f_1^p = (1-\rho)\, f_2^{\rho+1} f_3^\sigma f_1^p \\ &[e_1,\ f_2^\rho f_3^\sigma f_1^p] = \sigma(1-\rho)(-1)^\sigma f_2^{\rho+1} f_3^{\sigma-1} f_1^p q^{h_1-2p+1} + [p] f_2^\rho f_3^\sigma f_1^{p-1} [h_1 - p + 1] \\ &e_2\, f_2^\rho f_3^\sigma f_1^p - (-1)^{\rho+\sigma} f_2^\rho f_3^\sigma f_1^p\, e_2 \\ &\qquad = \rho\, f_2^{\rho-1} f_3^\sigma f_1^p [h_2 + p + \sigma] + \sigma(-1)^\rho f_2^\rho f_3^{\sigma-1} f_1^{p+1} q^{-h_2-p} \end{aligned} \tag{B1}$$

where $(p,\ \rho,\ \sigma) \in \mathbb{N} \times \{0,\ 1\} \times \{0,\ 1\}$.

Typical nilpotent modules

$$\begin{aligned} k_1\, w_{\rho,\sigma,p} &= \lambda_1 q^{\rho-\sigma-2p}\, w_{\rho,\sigma,p} \\ k_2\, w_{\rho,\sigma,p} &= \lambda_2 q^{\sigma+p}\, w_{\rho,\sigma,p} \\ f_1\, w_{\rho,\sigma,p} &= q^{\sigma-\rho} w_{\rho,\sigma,p+1} - \rho(1-\sigma) q^{-\rho} w_{\rho-1,\sigma+1,p} \\ f_2\, w_{\rho,\sigma,p} &= (1-\rho)\, w_{\rho+1,\sigma,p} \\ e_1\, w_{\rho,\sigma,p} &= -\sigma(1-\rho)\lambda_1 q^{-2p+1} w_{\rho+1,\sigma-1,p} + [p][\mu_1 - p + 1] w_{\rho,\sigma,p-1} \\ e_2\, w_{\rho,\sigma,p} &= \rho[\mu_2 + p + \sigma]\, w_{\rho-1,\sigma,p} + \sigma(-1)^\rho \lambda_2^{-1} q^{-p} w_{\rho,\sigma-1,p+1} \end{aligned} \tag{B2}$$

with $(p,\ \rho,\ \sigma) \in \{0, \ldots, N-1\} \times \{0,\ 1\} \times \{0,\ 1\}$ in the left-hand side and, by convention, $w_{\rho,\sigma,N} = 0$ in the right-hand side. For *type* $\mathcal{A}$ modules, $q^{\mu_1} \equiv \lambda_1 = \omega q^{N-1}$. For *type* $\mathcal{B}$ nilpotent modules, $N = l'$ and $q^{\mu_1} \equiv \lambda_1$ is free.

Atypical nilpotent modules: the $[\mu_2] = 0$ case

$$\begin{aligned} k_1\, w_{\sigma,p} &= \lambda_1 q^{-\sigma-2p}\, w_{\sigma,p} \\ k_2\, w_{\sigma,p} &= \varepsilon q^{\sigma+p}\, w_{\sigma,p} \\ f_1\, w_{\sigma,p} &= q^\sigma\, w_{\sigma,p+1} \\ f_2\, w_{\sigma,p} &= (1-\sigma) q^{p-1} [p]\, w_{\sigma+1,p-1} \\ e_1\, w_{\sigma,p} &= q^{-\sigma} [p][\mu_1 + 1 - p - \sigma] w_{\sigma,p-1} \\ e_2\, w_{\sigma,p} &= \sigma \varepsilon q^{-p} w_{\sigma-1,p+1} \end{aligned} \tag{B3}$$

where $\sigma \in \{0,\ 1\}$. For *type* $\mathcal{A}$ representations, $p \in \{0, \ldots, N-1-\sigma\}$ and the dimension is $2\,N - 1$. For *type* $\mathcal{B}$ representations, $p \in \{0, \ldots, l'-1\}$ and the dimension is $2\,l'$.

Atypical nilpotent modules: the $[\mu_1+\mu_2+1]=0$ *case*

$$
\begin{aligned}
k_1\, w_{\sigma,p} &= \lambda_1 q^{-\sigma-2p}\, w_{\sigma,p} \\
k_2\, w_{\sigma,p} &= \varepsilon\lambda_1^{-1} q^{\sigma+p-1}\, w_{\sigma,p} \\
f_1\, w_{\sigma,p} &= q^{\sigma}\, w_{\sigma,p+1} \\
f_2\, w_{\sigma,p} &= -(1-\sigma)\lambda_1^{-1} q^{p-2}[\mu_1-p+1]\, w_{\sigma+1,p-1} \\
e_1\, w_{\sigma,p} &= q^{-\sigma}[p+\sigma][\mu_1+1-p] w_{\sigma,p-1} \\
e_2\, w_{\sigma,p} &= \sigma\varepsilon\lambda_1 q^{-p+1} w_{\sigma-1,p+1}
\end{aligned}
\tag{B4}
$$

where $\sigma \in \{0,\ 1\}$. For *type* $\mathcal{A}$ representations, $p \in \{-\sigma, \ldots, N-1\}$ and the dimension is $2N+1$. For *type* $\mathcal{B}$ representations, $p \in \{0, \ldots, l'-1\}$ and the dimension is $2\,l'$.

Typical periodic modules

The actions of the generators e_1, e_2, f_1 and f_2 on a typical periodic M module are given by

$$
\begin{aligned}
k_1\, w_{\rho,\sigma,p} &= \lambda_1 q^{\rho-\sigma-2p}\, w_{\rho,\sigma,p} \\
k_2\, w_{\rho,\sigma,p} &= \lambda_2 q^{\sigma+p}\, w_{\rho,\sigma,p} \\
f_1\, w_{\rho,\sigma,p} &= \varphi q^{\sigma-\rho} w_{\rho,\sigma,p+1} - \varphi\rho(1-\sigma) q^{-\rho} w_{\rho-1,\sigma+1,p} \\
f_2\, w_{\rho,\sigma,p} &= (1-\rho)\, w_{\rho+1,\sigma,p} \\
e_1\, w_{\rho,\sigma,p} &= -\varphi^{-1}\sigma(1-\rho)\lambda_1 q^{-2p+1} w_{\rho+1,\sigma-1,p} + \varphi^{-1}\left([p][\mu_1-p+1]+\beta\right) w_{\rho,\sigma,p-1} \\
e_2\, w_{\rho,\sigma,p} &= \rho[\mu_2+p+\sigma]\, w_{\rho-1,\sigma,p} + \sigma(-1)^{\rho}\lambda_2^{-1} q^{-p} w_{\rho,\sigma-1,p+1}
\end{aligned}
\tag{B5}
$$

with $(\rho,\sigma) \in \{0,\ 1\}^2$ and $p \in \{0, \ldots, l-1\}$.

Atypical periodic modules

$$
\begin{aligned}
k_1\, w_{\sigma,p} &= \lambda_1 q^{-\sigma-2p}\, \tilde{w}_{\sigma,p} \\
k_2\, w_{\sigma,p} &= \lambda_2 q^{\sigma+p}\, \tilde{w}_{\sigma,p} \\
f_1\, w_{\sigma,p} &= \varphi q^{\sigma} w_{\sigma,p+1} \\
f_2\, w_{\sigma,p} &= (1-\sigma)\lambda_2 q^{p-1}\, [\mu_2+p]\, \tilde{w}_{\sigma+1,p-1} \\
e_1\, w_{\sigma,p} &= \varphi^{-1} q^{-\sigma}[p+\mu_2][\mu_1+\mu_2-p+1-\sigma] w_{\sigma,p-1} \\
e_2\, w_{\sigma,p} &= \sigma\lambda_2^{-1} q^{-p} w_{\sigma-1,p+1}
\end{aligned}
\tag{B6}
$$

with $\sigma \in \{0,\ 1\}$ and $p \in \{0, \ldots, l-1\}$.

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